
Definition: Let $a_i \in \mathbb{R}$.

$$a_1 + a_2 + ... + a_n + ... = \sum_{k=1}^{\infty} a_k$$

is called an **infinite series**.

1. a_i are called the **terms** of the series
2. $s_n = a_1 + ... + a_n$ are called the **partial sums** of the series

Definition:
If $\exists s$ such that $\lim_{n\to\infty} s_n = s$, then we say that the series is **convergent**.
If the s_n do not tend to a limit, we say that the series is **divergent**.

Theorem 1. If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Does the above theorem imply that $\sum_{n=1}^{\infty} \frac{1}{n}$ converges?

Theorem 2. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be given series and let $c \neq 0$ be a constant. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then $\sum_{n=1}^{\infty}(a_n + b_n)$, $\sum_{n=1}^{\infty}(a_n - b_n)$, and $\sum_{n=1}^{\infty} ca_n$ are convergent series. Moreover,

$$\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$

Changing the Order of Summation: The terms of a finite sum can be added in any order without changing the result, since addition is commutative.

$$a + b + c + d = d + a + c + b$$

Can we do the same with infinite series?

In general not!

For example $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ converges. However, the rearrangement

$$1 + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{9}} + \frac{1}{\sqrt{11}} - \frac{1}{\sqrt{6}} + ...$$

diverges.

In fact, the terms in the infinite sum can be rearranged to add up to **any** real number!

Example Find a general expression for the terms of the series.

1. $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + ...$

2. $2 + 10 + 50 + 250 + 1250 + ...$

3. $-3 + \frac{3}{2} - \frac{3}{4} + \frac{3}{8} - \frac{3}{16} + ...$

Geometric Series: A series of the form

$$a + ar + ar^2 + ar^3 + ... + ar^n + ...$$

is called a **geometric series**. The number r is the **common ratio**.

Theorem 3. A geometric series with $a \neq 0$ converges if $|r| < 1$ and diverges if $|r| \geq 1$. In the convergent case, we have

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

proof: Show that $s_{n-1} = a + ar + ... + ar^{n-1} = \frac{a}{1-r} - \frac{ar^n}{1-r}$.